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On singular, functional, nonsmooth and implicit phi-Laplacian initial and boundary value problems

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Abstract

In this paper we apply fixed point results for mappings in partially ordered spaces to derive existence results for extremal solutions of phi-Laplacian initial and boundary value problems. The considered problems can be singular, functional, discontinuous, nonlocal and implicit. Concrete examples are also solved.

Keywords: Initial value problem; Boundary value problem; Implicit; Singular; Functional; Nonlocal; Phi-Laplacian; Discontinuous; Solution; Least; Greatest

1. Introduction

In this paper we apply fixed point results presented in [4,6] for mappings in partially ordered spaces to derive existence results for first- and second-order differential equations. The considered problems include many kinds of special types. For instance,

– differential equations and initial/boundary conditions may be implicit;

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– differential operators of differential equations may be singular;
– differential equations and initial or boundary conditions may depend functionally on the unknown function and its derivative;
– differential equations and initial or boundary conditions may contain discontinuous nonlinearities;
– problems on infinite intervals are also included.

Concrete examples are also presented and solved to illustrate the obtained results.

2. Existence results for first-order implicit initial value problems

In this section we study the first-order implicit initial value problem (IVP)

$$\begin{cases} Lu(t) := \frac{d}{dt}(p(t)\phi(u(t))) = f(t, u, Lu) \\ \quad \text{for almost every (a.e.) } t \in J := (a, b), \\ \lim_{t\to a+} p(t)\phi(u(t)) = c(u, Lu), \end{cases} \tag{2.1}$$

where $-\infty \leqslant a < b \leqslant \infty$, $p \in L^1_{\mathrm{loc}}(J)$, $\phi : I \to \mathbb{R}$, $f : J \times L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J) \to \mathbb{R}$ and $c : L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J) \to \mathbb{R}$.

We are looking for solutions of (2.1) from the set

$$S := \{u \in L^1_{\mathrm{loc}}(J) \mid p \cdot (\phi \circ u) \text{ is locally absolutely continuous}\}. \tag{2.2}$$

Denote

$$X := \left\{h \in L^1_{\mathrm{loc}}(J) \,\middle|\, \int_{a+}^{s} h(t)\,dt = \lim_{r\downarrow a} \int_{r}^{s} h(t)\,dt \text{ is finite for some } s \in J\right\}. \tag{2.3}$$

Assuming that $L^1_{\mathrm{loc}}(J)$, X and S are ordered a.e. pointwise, we shall show that the IVP (2.1) has extremal solutions in S if the functions p, ϕ, f and c satisfy the following hypotheses:

(ϕ) ϕ is an increasing homeomorphism from an open interval I of $\mathbb{R}$ onto $\mathbb{R}$.
($p\phi$) p is a.e. positive-valued and $\phi^{-1}\big(\frac{K}{p(\cdot)}\big) \in L^1_{\mathrm{loc}}(J)$ for all $K \in \mathbb{R}$.
(fa) $f(\cdot, u, v)$ is Lebesgue measurable and $h_- \leqslant f(\cdot, u, v) \leqslant h_+$ for all $u, v \in L^1_{\mathrm{loc}}(J)$ and for some $h_\pm \in X$.
(fb) There exists a $\lambda \geqslant 0$ such that $f(\cdot, u_1, v_1) + \lambda v_1 \leqslant f(\cdot, u_2, v_2) + \lambda v_2$ whenever $u_i, v_i \in L^1_{\mathrm{loc}}(J)$, $i = 1, 2$, $u_1 \leqslant u_2$ and $v_1 \leqslant v_2$.
(c) $c_\pm \in \mathbb{R}$ and $c_- \leqslant c(u_1, v_1) \leqslant c(u_2, v_2) \leqslant c_+$ whenever $u_i, v_i \in L^1_{\mathrm{loc}}(J)$, $i = 1, 2$, $u_1 \leqslant u_2$ and $v_1 \leqslant v_2$.

We shall first convert the IVP (2.1) to a system of two equations.

Lemma 2.1. *Assume that the hypotheses* (ϕ) *and* ($p\phi$) *hold, and that* $f(\cdot, u, v) \in X$ *for all* $u, v \in L^1_{\mathrm{loc}}(J)$. *Then* u *is a solution of the IVP* (2.1) *in* S *if and only if* $(u, Lu) = (u, v)$, *where* (u, v) *is a solution of the system*

$$\begin{cases} u(t) = \phi^{-1}\big(\frac{1}{p(t)}[c(u,v) + \int_{a+}^{t} v(s)\,ds]\big), & t \in J, \\ v(t) = f(t, u, v) & \text{for a.e. } t \in J, \end{cases} \tag{2.4}$$

in $L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J)$.

Proof. Assume that u is a solution of (2.1) in S. Denoting

$$v(t) = Lu(t) = \frac{d}{dt}\big(p(t)\phi\big(u(t)\big)\big), \quad t \in J, \tag{2.5}$$

the definition (2.2) of S and (2.5) ensure that

$$\int_r^s v(t)\,dt = \int_r^s \frac{d}{dt}\big(p(t)\phi\big(u(t)\big)\big)\,dt = p(s)\phi\big(u(s)\big) - p(r)\phi\big(u(r)\big), \quad a < r \leqslant s < b.$$

This result and the initial condition of (2.1) imply that the first equation of (2.4) holds. The validity of the second equation of (2.4) is a consequence of the differential equation of (2.1) and the definition (2.5) of v.

Conversely, let (u, v) be a solution of the system (2.4) in $L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J)$. According to (2.4) we have

$$p(t)\phi\big(u(t)\big) = c(u, v) + \int_{a+}^{t} v(s)\,ds, \quad t \in J. \tag{2.6}$$

This equation implies that $u \in S$, and by differentiation we obtain from (2.6) that

$$v(t) = \frac{d}{dt} p(t)\phi\big(u(t)\big) = Lu(t) \quad \text{for a.e. } t \in J.$$

This result, Eq. (2.6) and the second equation of (2.4) imply that u is a solution of the IVP (2.1). □

The following fixed point result is a consequence of [4, Theorem A.2.1], or [6, Theorem 1.2.1 and Proposition 1.2.1].

Lemma 2.2. *Given a partially ordered set* $P = (P, \leqslant)$ *and its order interval* $[x_-, x_+] = \{x \in P \mid x_- \leqslant x \leqslant x_+\}$, *assume that* $G : [x_-, x_+] \to [x_-, x_+]$ *is increasing, i.e.,* $Gx \leqslant Gy$ *whenever* $x_- \leqslant x \leqslant y \leqslant x_+$, *and that each well-ordered chain of the range* ran G *of* G *has a supremum in* P *and each inversely well-ordered chain of* ran G *has an infimum in* P. *Then* G *has least and greatest fixed points, and they are increasing with respect to* G.

In the application of Lemma 2.2 to the IVP (2.1) we use Lemma 2.1 and the following result.

Lemma 2.3. *Assume that W is a nonempty subset of $L^1_{\mathrm{loc}}(J)$, $J = (a,b)$, $-\infty \leqslant a < b \leqslant \infty$, and that there exist functions $u_\pm \in L^1_{\mathrm{loc}}(J)$, $i = 1, 2$, such that*

$$u_-(t) \leqslant u(t) \leqslant u_+(t) \quad \text{for all } u \in W \text{ and for a.e. } t \in J. \tag{2.7}$$

(a) *If W is well-ordered, it contains an increasing sequence which converges a.e. pointwise to* $\sup W$.
(b) *If W is inversely well-ordered, it contains a decreasing sequence which converges a.e. pointwise to* $\inf W$.

Proof. (a) Assume that W is well-ordered. Choose a sequence of finite and closed subintervals J_n, $n \in \mathbb{N}$, of J such that $J = \bigcup_{n=0}^\infty J_n$, and that $J_n \subset J_{n+1}$ for each $n \in \mathbb{N}$. The given assumptions ensure that for each $n \in \mathbb{N}$ the restrictions $u|_{J_n}$, $u \in W$, form a well-ordered and order-bounded chain W_n in $L^1(J_n)$, ordered a.e. pointwise. Consequently, by the proof of [5, Lemma 4.2], for each $n \in \mathbb{N}$,

$$v_n = \sup W_n$$

exists in $L^1(J_n)$, and there exists an increasing sequence $(u_n^k)_{k=0}^\infty$ of W and a null-set $Z_n \subset J_n$ such that

$$v_n(t) = \lim_{k\to\infty} u_n^k(t) = \sup_{k\in\mathbb{N}} u_n^k(t) \quad \text{for each } t \in J_n \setminus Z_n. \tag{2.8}$$

Defining $v_n(t) = 0$ for $t \in J \setminus J_n$, we obtain a sequence of Lebesgue measurable functions $v_n : J \to \mathbb{R}$. The sequence (v_n) is also increasing since $J_n \subset J_{n+1}$, $n \in \mathbb{N}$. It is also a.e. pointwise bounded by (2.7) and (2.8), whence

$$u^*(t) = \lim_{n\to\infty} v_n(t) = \sup_{n\in\mathbb{N}} v_n(t) \tag{2.9}$$

exists for a.e. $t \in J$. Defining $u^*(t) = 0$ for the remaining $t \in J$, we obtain a Lebesgue measurable function $u^* : J \to \mathbb{R}$. Denoting

$$u_n = \max\{u_j^n \mid 0 \leqslant j \leqslant n\}, \quad n \in \mathbb{N},$$

we obtain an increasing sequence (u_n) of W which satisfies

$$u_n^k(t) \leqslant u_n(t) \leqslant u^*(t)$$

for each $k = 0, \dots, n$ and $t \in J_n \setminus Z_n$. Moreover, by (2.7) the sets Z_n can be so chosen that $(u_n(t))_{n=0}^\infty$ is bounded and increasing for each $t \in J \setminus Z$, where $Z = \bigcup_{n=0}^\infty Z_n$. Thus

$$u(t) = \lim_{n\to\infty} u_n(t) = \sup_{n\in\mathbb{N}} u_n(t)$$

exists for each $t \in J \setminus Z$. The definitions of v_n and u imply that

$$v_n(t) \leqslant u(t) \leqslant u^*(t) \quad \text{for each } t \in J_n \setminus Z_n.$$

Thus

$$u^*(t) = \lim_{n\to\infty} v_n(t) \leqslant u(t) \leqslant u^*(t)$$

for a.e. $t \in J$. This result implies that $u = u^*$, whence $u_n(t) \to u^*(t)$ for a.e. $t \in J$. Since $(u_n)_{n=0}^{\infty}$ is a sequence of W, it follows from (2.7) that

$$u_-(t) \leqslant u^*(t) \leqslant u_+(t) \quad \text{for a.e. } t \in J.$$

This result and Lebesgue measurability of u^* imply that $u^* \in L^1_{\text{loc}}(J)$.

It remains to prove that $u^* = \sup W$. If $w \in W$, then $w|_{J_n} \leqslant v_n$, whence

$$w(t) \leqslant v_n(t) \leqslant u^*(t) \quad \text{for a.e. } t \in J_n \text{ and for each } n \in \mathbb{N}.$$

Thus $w \leqslant u^*$ for each $w \in W$, so that u^* is an upper bound of W. If $v \in L^1_{\text{loc}}(J)$ is another upper bound of W, then for each $w \in W$

$$w(t) \leqslant v(t) \quad \text{for a.e. } t \in J_n \text{ and for each } n \in \mathbb{N}.$$

Thus $w|_{J_n} \leqslant v|_{J_n}$ for all $n \in \mathbb{N}$ and $w \in W$, whence

$$v_n(t) \leqslant v(t) \quad \text{for a.e. } t \in J_n \text{ and for each } n \in \mathbb{N}.$$

This result and the definition (2.9) of u^* imply that $u^* \leqslant v$. Consequently, $u^* = \sup W$ in $L^1_{\text{loc}}(J)$.

(b) If W is inversely well-ordered, then $-W$, satisfies the hypotheses imposed on W in (a). Thus there exists an increasing sequence (u_n) in $-W$ such that $u_n \to u = \sup(-W)$ a.e. pointwise on J. Denoting $w_n = -u_n$, $n \in \mathbb{N}$, we obtain a decreasing sequence of W which converges a.e. pointwise to $-u = \inf W$. $\square$

Now we are ready to prove our main existence result for the IVP (2.1).

Theorem 2.1. *Assume that the hypotheses* (ϕ), ($p\phi$), (fa), (fb) *and* (c) *hold. Then the IVP* (2.1) *has least and greatest solutions in S, and they are increasing with respect to f and c.*

Proof. Assume that $P = L^1_{\text{loc}}(J) \times L^1_{\text{loc}}(J)$ is ordered componentwise. The relations

$$x_\pm(t) := \left(\phi^{-1}\left(\frac{1}{p(t)}\left[c_\pm + \int_{a+}^{t} h_\pm(s)\,ds\right]\right), h_\pm(t)\right), \quad t \in J, \tag{2.10}$$

define functions $x_\pm \in P$. If $(u, v) \in [x_-, x_+]$, then $v \in L^1_{\text{loc}}(J)$ and $h_- \leqslant v \leqslant h_+$. Hence it is easy to show that $v \in X$, because $h_\pm \in X$. Thus, by applying the given hypotheses, we see that the relations

$$\begin{aligned} G_1(u, v)(t) &= \phi^{-1}\left(\frac{1}{p(t)}\left[c(u, v) + \int_{a+}^{t} v(s)\,ds\right]\right), \\ G_2(u, v)(t) &= \frac{f(t, u, v) + \lambda v(t)}{1 + \lambda}, \end{aligned} \tag{2.11}$$

define an increasing mapping $G = (G_1, G_2) : [x_-, x_+] \to [x_-, x_+]$.

Let W be a well-ordered chain in ran G. The sets $W_1 = \{u \mid (u, v) \in W\}$ and $W_2 = \{v \mid (u, v) \in W\}$ are well-ordered and order-bounded chains in $L^1_{\text{loc}}(J)$. It then follows from Lemma 2.3 that $\sup W_1$ and $\sup W_2$ exist in $L^1_{\text{loc}}(J)$. Obviously, $(\sup W_1, \sup W_2)$ is a supremum of W in P. Similarly one can show that each inversely well-ordered chain of ran G has an infimum in P.

The above proof shows that the operator $G = (G_1, G_2)$ defined by (2.11) satisfies the hypotheses of Lemma 2.2, whence G has a least fixed point $x_* = (u_*, v_*)$ and a greatest fixed point $x^* = (u^*, v^*)$. It follows from (2.11) that (u_*, v_*) and (u^*, v^*) are solutions of the system (2.4). According to Lemma 2.1 u_* and u^* belong to S and are solutions of the IVP (2.1).

To prove that u_* and u^* are least and greatest of all solutions of (2.1) in S, let $u \in S$ be a solution of (2.1). In view of Lemma 2.1, $(u, v) = (u, Lu)$ is a solution of the system (2.4). Applying the hypotheses (fa) and (c) it is easy to show that $x = (u, v) \in [x_-, x_+]$, where $x_\pm$ are defined by (2.10). Thus $x = (u, v)$ is a fixed point of $G = (G_1, G_2) : [x_-, x_+] \to [x_-, x_+]$, defined by (2.11). Because $x_* = (u_*, v_*)$ and $x^* = (u^*, v^*)$ are least and greatest fixed points of G, then $(u_*, v_*) \leqslant (u, v) \leqslant (u^*, v^*)$. In particular, $u_* \leqslant u \leqslant u^*$, whence u_* and u^* are least and greatest of all solutions of the IVP (2.1).

The last assertion is an easy consequence of the last conclusion of Lemma 2.2 and the definition of G. $\square$

As a special case, we obtain an existence result for the IVP

$$\begin{cases} \frac{d}{dt}(p(t)\phi(u(t))) = g(t, u(t), \frac{d}{dt}(p(t)\phi(u(t)))) & \text{for a.e. } t \in J, \\ \lim_{t\to a+} p(t)\phi(u(t)) = c. \end{cases} \tag{2.12}$$

Proposition 2.1. *Let the hypotheses* (ϕ) *and* $(p\phi)$ *hold, and let* $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfy the following hypotheses*:

(ga) $g(\cdot, u(\cdot), v(\cdot))$ *is Lebesgue measurable and* $h_- \leqslant g(\cdot, u(\cdot), v(\cdot)) \leqslant h_+$ *for all* $u, v \in L^1_{\mathrm{loc}}(J)$ *and for some* $h_\pm \in X$.
(gb) $g(t, x, z) \leqslant g(t, y, w)$ *for a.e.* $t \in J$ *and whenever* $x \leqslant y$ *and* $z \leqslant w$ *in* $\mathbb{R}$.

Then the IVP (2.12) *has for each choice of* $c \in \mathbb{R}$ *least and greatest solutions in* S. *Moreover, these solutions are increasing with respect to* g *and* c.

Proof. If $c \in \mathbb{R}$, the IVP (2.12) is reduced to (2.1) when we define

$$\begin{cases} f(t, u, v) = g(t, u(t), v(t)), & t \in J,\ u, v \in L^1_{\mathrm{loc}}(J), \\ c(u, v) \equiv c, & u, v \in L^1_{\mathrm{loc}}(J). \end{cases} \tag{2.13}$$

The hypotheses (ga) and (gb) imply that f satisfies the hypotheses (fa) and (fb). The hypothesis (c) is also valid, whence (2.1), with f and c defined by (2.13), and hence also (2.12), has by Theorem 2.1 least and greatest solutions. The last assertion follows from the last assertion of Theorem 2.1. $\square$

If we replace the hypothesis (fa) by the following hypothesis:

(fc) $f(\cdot,u,v) \in X$ for all $u, v \in L^1_{\mathrm{loc}}(J)$, and there exist $x_\pm = (u_\pm, v_\pm) \in L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J)$ such that $x_- \leqslant x_+$, $x_- \leqslant Gx_-$ and $Gx_+ \leqslant x_+$, where $G = (G_1, G_2)$ is defined by (2.11),

we get the following result.

Corollary 2.1. *Assume that the hypotheses* (ϕ), ($p\phi$), (fb), (fc) *and* (c) *hold. Then the IVP* (2.1) *has least and greatest solutions in* $\{u \in S \mid u_- \leqslant u \leqslant u_+\}$.

Remarks 2.1. If $\lim_{t\to a+} p(t) = 0$, the differential operator $\frac{d}{dt}(p(t)\phi(u(t)))$ in (2.1) is singular. An example of a function ϕ with property (ϕ) is

$$\phi(x) = \frac{x}{\sqrt{1-x^2}}, \quad x \in (-1,1),$$

arising in relativistic dynamics. In this case the operator $G = (G_1, G_2)$ given by (2.11) can be rewritten as

$$\begin{aligned} G_1(u,v)(t) &= \frac{c(u,v) + \int_{a+}^t v(s)\,ds}{\sqrt{p^2(t) + (c(u,v) + \int_{a+}^t v(s)\,ds)^2}}, \\ G_2(u,v)(t) &= \frac{f(t,u,v) + \lambda v(t)}{1+\lambda}. \end{aligned} \tag{2.14}$$

This formula shows that $-1 \leqslant G_1(u,v)(t) \leqslant 1$ whenever $G_1(u,v)(t)$ is defined. Thus we have the following result.

Proposition 2.2. *Assume that* $p \in L^1_{\mathrm{loc}}(J)$ *is positive-valued, that the hypotheses* (fa) *and* (fb) *hold, and that* $c(u_1,v_1) \leqslant c(u_2,v_2)$ *in* $\mathbb{R}$ *whenever* $u_1 \leqslant v_i$ *in* $L^1_{\mathrm{loc}}(J)$, $i = 1, 2$. *Then the IVP*

$$\begin{cases} Lu := \frac{d}{dt}\big(\frac{p(t)u(t)}{\sqrt{1-u(t)^2}}\big) = f(t,u,Lu) \quad \text{a.e. in } J = (a,b), \\ \lim_{t\to a+} \frac{p(t)u(t)}{\sqrt{1-u(t)^2}} = c(u,Lu) \end{cases} \tag{2.15}$$

has least and greatest solutions in S.

As a consequence of Proposition 2.2, we obtain an existence result also for a periodic boundary value problem.

Corollary 2.2. *Let* p *and* f *satisfy the hypotheses of Proposition* 2.2. *Then for each choice of* $t_1, t_2 \in J$, $t_1 < t_2$, *the periodic boundary value problem*

$$Lu := \frac{d}{dt}\left(\frac{p(t)u(t)}{\sqrt{1-u(t)^2}}\right) = f(t,u,Lu) \quad \text{a.e. in } [t_1,t_2], \qquad u(t_1) = u(t_2) \tag{2.16}$$

has least and greatest solutions.

Proof. The asserted result follows from Proposition 2.2 when we replace a by t_1 and $c(u, Lu)$ by $p(t_1)\frac{u(t_2)}{\sqrt{1-u(t_2)^2}}$ in (2.15). $\square$

Example 2.1. Choose $J = (0, \infty)$, and consider the IVP

$$\begin{cases} Lu(t) := \frac{d}{dt}\left(\frac{p(t)u(t)}{\sqrt{1-u(t)^2}}\right) \\ \qquad = h(t) + \left[K \tanh\left(q(t)\int_1^4 (u(s) + Lu(s))\,ds\right)\right]/K \quad \text{a.e. in } J, \\ \lim_{t\to 0+} \frac{p(t)u(t)}{\sqrt{1-u(t)^2}} = c \cdot \frac{p(1)u(1)}{\sqrt{1-u(1)^2}}, \end{cases} \tag{2.17}$$

where $p \in L^1_{\mathrm{loc}}(J)$, $p(t) > 0$ for $t \in J$, $q \in L^1_+(J)$, $h \in X$, $c \geqslant 0$, $K > 0$ and $[z]$ denotes the greatest integer $\leqslant z$. Problem (2.17) is of the form (2.15) with

$$c(u) = c \cdot \frac{p(1)u(1)}{\sqrt{1-u(1)^2}} \quad \text{and}$$

$$f(t, u, v) = h(t) + \left[K \tanh\left(q(t)\int_1^4 \big(u(s) + v(s)\big)\,ds\right)\right]\Big/K, \quad t \in J. \tag{2.18}$$

It is easy to see that the hypotheses of Corollary 2.2 hold, whence (2.12) has least and greatest solutions.

Remark 2.2. If $h(t) = \frac{1}{t}\sin\frac{1}{t}$, $t \in J = (0, \infty)$, then h and the function $f(\cdot, u, v)$ defined by (2.18) belong to X, but not to $L^1((0, T))$ for any $T > 0$.

Example 2.2. The singular IVP

$$\begin{cases} Lu(t) := \frac{d}{dt}\left(\frac{tu(t)}{\sqrt{1-u(t)^2}}\right) \\ \qquad = \frac{100(t-1)+[1000\tanh(\int_1^4 (u(s)+Lu(s)/4)\,ds)]}{1000} \quad \text{a.e. in } (0, \infty), \\ \lim_{t\to 0+} \frac{tu(t)}{\sqrt{1-u(t)^2}} = 0, \end{cases} \tag{2.19}$$

is a special case of (2.17) when $p(t) = t$, $q(t) \equiv \frac{1}{4}$, $h(t) = \frac{t-1}{10}$, $c = 0$ and $K = 1000$. Thus (2.19) has extremal solutions. To determine them, notice first that we can choose $c_\pm = 0$ and $h_\pm(t) = \frac{t-1}{10} \pm 1$ in (2.10). Thus the functions $x_\pm$ defined by (2.10) can be calculated, and one obtains

$$x_-(t) = \left(\frac{t-22}{\sqrt{t^2 - 44t + 887}}, \frac{t}{10} - \frac{11}{10}\right),$$
$$x_+(t) = \left(\frac{t+18}{\sqrt{t^2 + 36t + 724}}, \frac{t}{10} - \frac{9}{10}\right).$$

In this case the chains needed in the proof of Corollary 2.2 are reduced to ordinary iteration sequences $(G^n x_\pm)$, where $G = (G_1, G_2)$ is defined by (2.14). Calculating the iterations $G^n x_-$, or equivalently, the successive approximations

$$\begin{cases} u_n(t) = \frac{\int_{a+}^t v_{n-1}(s)\,ds}{\sqrt{t^2+(\int_{a+}^t v_{n-1}(s)\,ds)^2}}, & t \in J, \\ v_n(t) = \frac{100(t-1)+[\tanh(1000\int_1^4 (u_{n-1}(s)+v_{n-1}(s))\,ds)]}{1000} & \text{a.e. in } (0,\infty), \\ u_0(t) = \frac{t-22}{\sqrt{t^2-44t+884}}, \quad v_0(t) = \frac{t}{10} - \frac{461}{500}, \end{cases} \tag{2.20}$$

it turns out that $G^5 x_- = G^6 x_-$. Thus $u_* = G_1^5 x_-$ is a least solution of (2.19). Similarly, one can show that $u^* = G_1^{14} x_+ = G^{15} x_+$ is a greatest solution of (2.19). The exact expressions of these solutions are

$$u_*(t) = \frac{25t-373}{\sqrt{625t^2-18650t+389129}}, \qquad u^*(t) = \frac{25t+372}{\sqrt{625t^2+18600t+388384}}.$$

3. Existence results for second-order implicit initial value problems

Next we study the second-order implicit initial value problem (IVP),

$$\begin{cases} Lu(t) := \frac{d}{dt}(p(t)\phi(u'(t))) = f(t,u,u',Lu) & \text{for a.e. } t \in J := (a,b), \\ \lim_{t\to a+} p(t)\phi(u'(t)) = c(u,u',Lu), & \lim_{t\to a+} u(t) = d(u,u',Lu), \end{cases} \tag{3.1}$$

where $-\infty \leqslant a < b \leqslant \infty$, $p \in L^1_{\mathrm{loc}}(J)$, $\phi : I \to \mathbb{R}$, $f : J \times L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J) \to \mathbb{R}$ and $c, d : L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J) \to \mathbb{R}$.

We are now looking for solutions of (3.1) from the set

$$Y := \{u : J \to \mathbb{R} \mid u \text{ and } p \cdot (\phi \circ u') \text{ are locally absolutely continuous}\}. \tag{3.2}$$

Denote, as in Section 2,

$$X := \left\{ h \in L^1_{\mathrm{loc}}(J) \;\middle|\; \int_{a+}^{s} h(t)\,dt = \lim_{r \downarrow a} \int_r^s h(t)\,dt \text{ is finite for some } s \in J \right\}. \tag{3.3}$$

Assuming that $L^1_{\mathrm{loc}}(J)$ and X are ordered a.e. pointwise, and that Y is ordered pointwise, we shall show that the IVP (3.1) has extremal solutions in Y if the functions p, ϕ, f, c and d satisfy the following hypotheses:

(ϕ) ϕ is an increasing homeomorphism from an open interval I of $\mathbb{R}$ onto $\mathbb{R}$.
($p\phi$) p is a.e. positive-valued, and $\phi^{-1}\left(\frac{K}{p(\cdot)}\right) \in X$ for all $K \in \mathbb{R}$.
(f0) $f(\cdot,u,v,w)$ is Lebesgue measurable and $X \ni h_- \leqslant f(\cdot,u,v,w) \leqslant h_+ \in X$ for all $u, v, w \in L^1_{\mathrm{loc}}(J)$.
(f1) There exists a $\lambda \geqslant 0$ such that $f(\cdot,u_1,v_1,w_1) + \lambda w_1 \leqslant f(\cdot,u_2,v_2,w_2) + \lambda w_2$ whenever $u_i, v_i, w_i \in L^1_{\mathrm{loc}}(J)$, $i = 1, 2$, $u_1 \leqslant u_2$, $v_1 \leqslant v_2$ and $w_1 \leqslant w_2$.

(c0) $c_\pm \in \mathbb{R}$, and $c_- \leqslant c(u_1, v_1, w_1) \leqslant c(u_2, v_2, w_2) \leqslant c_+$ whenever $u_i, v_i, w_i \in L^1_{\mathrm{loc}}(J)$, $i = 1, 2$, $u_1 \leqslant u_2$, $v_1 \leqslant v_2$ and $w_1 \leqslant w_2$.

(d0) $d_\pm \in \mathbb{R}$, and $d_- \leqslant d(u_1, v_1, w_1) \leqslant d(u_2, v_2, w_2) \leqslant d_+$ whenever $u_i, v_i, w_i \in L^1_{\mathrm{loc}}(J)$, $i = 1, 2$, $u_1 \leqslant u_2$, $v_1 \leqslant v_2$ and $w_1 \leqslant w_2$.

Our first task is to convert the IVP (3.1) to a system of equations which do not contain derivatives.

Lemma 3.1. *Assume that the hypotheses* (ϕ) *and* ($p\phi$) *hold, and that* $f(\cdot, u, v, w) \in X$ *for all* $u, v, w \in L^1_{\mathrm{loc}}(J)$. *Then* u *is a solution of the IVP* (3.1) *in* Y *if and only if* $(u, u', Lu) = (u, v, w)$, *where* (u, v, w) *is a solution of the system*

$$\begin{cases} u(t) = d(u, v, w) + \int_{a+}^t v(s)\,ds, & t \in J, \\ v(t) = \phi^{-1}\left(\frac{1}{p(t)}\left[c(u, v, w) + \int_{a+}^t w(s)\,ds\right]\right), & t \in J, \\ w(t) = f(t, u, v, w) & \textit{for a.e. } t \in J \end{cases} \tag{3.4}$$

in $L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J)$.

Proof. Assume that u is a solution of (3.1) in Y, and denote

$$w(t) = Lu(t) = \frac{d}{dt}\big(p(t)\phi\big(u'(t)\big)\big), \qquad v(t) = u'(t), \quad t \in J. \tag{3.5}$$

The differential equation and the second initial condition of (3.1), the definition (3.2) of Y and notations (3.5) ensure that first and third equations of (3.4) hold, and that

$$\int_r^s w(t)\,dt = \int_r^s \frac{d}{dt}\big(p(t)\phi\big(v(t)\big)\big)\,dt = p(s)\phi\big(v(s)\big) - p(r)\phi\big(v(r)\big),$$
$$a < r \leqslant s < b.$$

This result and the first initial condition of (3.1) imply that the second equation of (3.4) holds.

Conversely, let (u, v, w) be a solution of the system (3.4) in $L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J)$. The first equation of (3.4) implies that $v = u'$, that u is locally absolutely continuous, and that the second initial condition of (3.1) holds. Since $v = u'$, it follows from the second equation of (3.4) that

$$p(t)\phi\big(u'(t)\big) = c(u, u', w) + \int_{a+}^t w(s)\,ds, \quad t \in J. \tag{3.6}$$

This equation implies that $p \cdot (\phi \circ u')$ is locally absolutely continuous and thus $u \in Y$. By differentiation we obtain from (3.6) that

$$w(t) = \frac{d}{dt} p(t)\phi\big(u'(t)\big) = Lu(t) \quad \text{for a.e. } t \in J. \tag{3.7}$$

This result and (3.6) imply that the first initial condition of (3.1) holds. The validity of the differential equation of (3.1) is a consequence of the third equation of (3.4), Eq. (3.7), and the fact that $v = u'$. □

Now we are ready to prove our main existence result for the IVP (3.1).

Theorem 3.1. *Assume that the hypotheses* (ϕ), ($p\phi$), (f0), (f1), (c0) *and* (d0) *hold. Then the IVP* (3.1) *has least and greatest solutions in Y, and they are increasing with respect to f, c and d.*

Proof. Assume that $P = L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J)$ is ordered componentwise. The relations

$$x_\pm(t) := \left(d_\pm + \int_{a+}^{t} \phi^{-1}\left(\frac{1}{p(s)} \left[c_\pm + \int_{a+}^{s} h_\pm(\tau)\, d\tau \right] \right) ds, \right.$$
$$\left. \phi^{-1}\left(\frac{1}{p(t)} \left[c_\pm + \int_{a+}^{t} h_\pm(s)\, ds \right] \right), h_\pm(t) \right) \tag{3.8}$$

define functions $x_\pm \in P$. If $(u, v, w) \in [x_-, x_+]$, then $w \in [h_-, h_+]$, whence $w \in X$. Hence, it is easy to show, by applying the given hypotheses, that the relations

$$\begin{cases} G_1(u, v, w)(t) = d(u, v, w) + \int_{a+}^{t} v(s)\, ds, & t \in J, \\ G_2(u, v, w)(t) = \phi^{-1}\left(\frac{1}{p(t)} \left[c(u, v, w) + \int_{a+}^{t} w(s)\, ds \right] \right), & t \in J, \\ G_3(u, v, w)(t) = \frac{f(t, u, v, w) + \lambda w(t)}{1 + \lambda}, & t \in J, \end{cases} \tag{3.9}$$

define an increasing mapping $G = (G_1, G_2, G_3) : [x_-, x_+] \to [x_-, x_+]$.

Let W be a well-ordered chain in ran G. The sets $W_1 = \{u \mid (u, v, w) \in W\}$, $W_2 = \{v \mid (u, v, w) \in W\}$ and $W_3 = \{w \mid (u, v, w) \in W\}$ are well-ordered and order-bounded chains in $L^1_{\mathrm{loc}}(J)$. It then follows from Lemma 2.3 that the supremums of W_1, W_2 and W_3 exist in $L^1_{\mathrm{loc}}(J)$. Obviously, $(\sup W_1, \sup W_2, \sup W_3)$ is a supremum of W in P. Similarly one can show that each inversely well-ordered chain of ran G has an infimum in P.

The above proof shows that the operator $G = (G_1, G_2, G_3)$ defined by (3.9) satisfies the hypotheses of Lemma 2.2, whence G has a least fixed point $x_* = (u_*, v_*, w_*)$ and a greatest fixed point $x^* = (u^*, v^*, w^*)$. It follows from (3.9) that (u_*, v_*, w_*) and (u^*, v^*, w^*) are solutions of the system (3.4). According to Lemma 3.1, u_* and u^* belong to Y and are solutions of the IVP (3.1).

To prove that u_* and u^* are least and greatest of all solutions of (3.1) in Y, let $u \in Y$ be a solution of (3.1). In view of Lemma 3.1, $(u, v, w) = (u, u', Lu)$ is a solution of the system (3.4). Applying the hypotheses (f0), (c0) and (d0) it is easy to show that $x = (u, v, w) \in [x_-, x_+]$, where $x_\pm$ are defined by (3.8). Thus $x = (u, v, w)$ is a fixed point of $G = (G_1, G_2, G_3) : [x_-, x_+] \to [x_-, x_+]$, defined by (3.9). Because $x_* = (u_*, v_*, w_*)$ and $x^* = (u^*, v^*, w^*)$ are least and greatest fixed points of G, then $(u_*, v_*, w_*) \leqslant (u, v, w) \leqslant (u^*, v^*, w^*)$. In particular, $u_* \leqslant u \leqslant u^*$, whence u_* and u^* are least and greatest of all solutions of the IVP (3.1).

The last assertion is an easy consequence of the last conclusion of Lemma 2.2 and the definition (3.9) of $G = (G_1, G_2, G_3)$. $\square$

If we replace the hypothesis (f0) by the following hypothesis:

(f2) $f(\cdot, u, v, w) \in X$ for all $u, v, w \in L^1_{\mathrm{loc}}(J)$, and there exist $x_\pm = (u_\pm, v_\pm, w_\pm) \in P = L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J)$ such that $x_- \leqslant x_+$, $x_- \leqslant Gx_-$ and $Gx_+ \leqslant x_+$, where $G = (G_1, G_2, G_3)$ is defined by (3.9),

we get the following result.

Proposition 3.1. *Assume that the hypotheses* (ϕ), ($p\phi$), (f1), (f2), (c0) *and* (d0) *hold. Then the IVP* (3.1) *has a least and a greatest solution in* $\{u \in Y \mid u_+ \leqslant u \leqslant u_+\}$.

As a special case, we obtain an existence result for the IVP,

$$\begin{cases} \frac{d}{dt}(p(t)\phi(u'(t))) = g\big(t, u(t), u'(t), \frac{d}{dt}(p(t)\phi(u'(t)))\big) & \text{for a.e. } t \in J, \\ \lim_{t\to a+} p(t)\phi(u'(t)) = c, \quad \lim_{t\to a+} u(t) = d. \end{cases} \tag{3.10}$$

Corollary 3.1. *Let the hypotheses* (ϕ) *and* ($p\phi$) *hold, and let* $g : J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfy the following hypotheses*:

(g0) *$g(\cdot, u(\cdot), v(\cdot), w(\cdot))$ is Lebesgue measurable and $h_- \leqslant g(\cdot, u(\cdot), v(\cdot), w(\cdot)) \leqslant h_+$ for all $u, v, w \in L^1_{\mathrm{loc}}(J)$ and for some $h_\pm \in X$.*

(g1) *There exists a $\lambda \geqslant 0$ such that $g(t, x_1, x_2, x_3) + \lambda x_3 \leqslant g(t, y_1, y_2, y_3) + \lambda y_3$ for a.e. $t \in J$ and whenever $x_1 \leqslant y_i$ in $\mathbb{R}$, $i = 1, 2, 3$.*

Then the IVP (3.10) *has for each choice of c, $d \in \mathbb{R}$ least and greatest solutions in Y. Moreover, these solutions are increasing with respect to g, c and d.*

Proof. If c, $d \in \mathbb{R}$, the IVP (3.10) is reduced to (3.1) when we define

$$\begin{cases} f(t, u, v, w) = g(t, u(t), v(t), w(t)), & t \in J,\ u, v, w \in L^1_{\mathrm{loc}}(J), \\ c(u, v, w) \equiv c, \quad d(u, v, w) \equiv d,\ u, v, w \in L^1_{\mathrm{loc}}(J). \end{cases}$$

The hypotheses (g0) and (g1) imply that f satisfies the hypotheses (f0) and (f1). The hypotheses (c0) and (d0) are also valid, whence (3.1), with f, c and d defined above, and hence also (3.10), has by Theorem 3.1 least and greatest solutions. The last assertion follows from the last assertion of Theorem 3.1. $\square$

Remarks 3.1. If $\lim_{t\to a+} p(t) = 0$, the differential operator $\frac{d}{dt}(p(t)\phi(u'(t)))$ in (3.1) is a singular phi-Laplacian operator. A special case of it is the p-Laplacian operator with

$$\phi(x) = |x|^{p-2}x, \quad x \in (-\infty, \infty) \text{ and } 1 < p < 2. \tag{3.11}$$

Another example of ϕ is

$$\phi(x) = \frac{x}{\sqrt{1 - x^2}}, \quad x \in (-1, 1).$$

In this case the operator G given by (3.9) can be rewritten as

$$\begin{cases} G_1(u,v,w)(t) = d(u,v,w) + \int_{a+}^t v(s)\,ds, & t \in J, \\ G_2(u,v,w)(t) = \frac{c(u,v,w)+\int_{a+}^t w(s)\,ds}{\sqrt{p^2(t)+(c(u,v,w)+\int_{a+}^t w(s)\,ds)^2}}, & t \in J, \\ G_3(u,v,w)(t) = \frac{f(t,u,v,w)+\lambda w(t)}{1+\lambda}, & t \in J. \end{cases} \tag{3.12}$$

This formula shows that $-1 \leqslant G_2(u,v,w)(t) \leqslant 1$ whenever $G_2(u,v,w)$ is defined. Thus we have the following result.

Corollary 3.2. *If $-\infty < a < b \leqslant \infty$, the IVP*

$$\begin{cases} Lu := \frac{d}{dt}\left(\frac{p(t)u'(t)}{\sqrt{1-u'(t)^2}}\right) = f(t,u,u',Lu) \quad \text{a.e. in } J = (a,b), \\ \lim_{t\to a+} \frac{p(t)u'(t)}{\sqrt{1-u'(t)^2}} = c(u,u',Lu), \quad u(a) = d(u,u',Lu), \end{cases} \tag{3.13}$$

has least and greatest solutions if $p \in L^1_{\mathrm{loc}}(J)$ is positive-valued, if the hypotheses (f0), (f1) *and* (d0) *hold, and if $c(u_1,v_1,w_1) \leqslant c(u_2,v_2,v_2)$ in $\mathbb{R}$ whenever $u_1 \leqslant u_2$, $v_1 \leqslant v_2$ and $w_1 \leqslant w_2$ in $L^1_{\mathrm{loc}}(J)$.*

Example 3.1. Choose $J = (0,\infty)$ and consider the IVP,

$$\begin{cases} Lu(t) := \frac{d}{dt}\left(\frac{p(t)u'(t)}{\sqrt{1-u'(t)^2}}\right) = h(t) + \frac{[K\tanh(q(t)\int_1^2(u(s)+u'(s)+Lu(s))\,ds)]}{K} \quad \text{a.e. in } J, \\ \lim_{t\to 0+} \frac{p(t)u'(t)}{\sqrt{1-u'(t)^2}} = c\cdot u'(1), \quad u(0) = \frac{[k\tan^{-1}(u(1)+u'(1))]}{2k}, \end{cases} \tag{3.14}$$

where $p \in L^1_{\mathrm{loc}}(J)$, $p(t) > 0$ for $t \in J$, $q \in L^1_+(J)$, $h \in X$, $c \geqslant 0$, $K,\ k > 0$ and $[z]$ denotes the greatest integer $\leqslant z$. The problem (3.14) is of the form (3.13) with

$$\begin{cases} f(t,u,v,w) = h(t) + \left[K\tanh\left(q(t)\int_1^2(u(s)+v(s)+w(s))\,ds\right)\right]/K, \quad t \in J, \\ c(u,v,w) = c\cdot v(1), \quad d(u,v,w) = \frac{[k\tan^{-1}(u(1)+v(1))]}{2k}. \end{cases} \tag{3.15}$$

It is easy to see that the hypotheses of Corollary 3.2 hold, whence the IVP (3.14) has least and greatest solutions.

Example 3.2. The singular IVP

$$\begin{cases} Lu(t) := \frac{d}{dt}\left(\frac{tu'(t)}{\sqrt{1-u'(t)^2}}\right) \\ \qquad = \frac{t}{8} - \frac{1}{10} + \frac{[100\tanh(\frac{1}{4}\int_1^2(u(s)+u'(s)+Lu(s))\,ds)]}{100} \quad \text{a.e. in } (0,\infty), \\ \lim_{t\to 0+} \frac{tu'(t)}{\sqrt{1-u'(t)^2}} = 0, \quad u(0) = \frac{1000\tan^{-1}(u(1)+u'(1))}{2000} \end{cases} \tag{3.16}$$

is a special case of (3.14) when $p(t) = t$, $q(t) \equiv \frac{1}{4}$, $h(t) = \frac{t}{8} - \frac{1}{10}$, $K = 100$, $k = 1000$ and $c = 0$. Thus (3.16) has extremal solutions. The functions $x_\pm$ defined by (3.8) with $d_\pm = \pm 1$, $c_\pm = 0$ and $h_\pm(t) = \frac{t}{8} - \frac{1}{10} \pm 1$ can be calculated, and one obtains

$$\begin{cases} x_-(t) = \left(-1 - \frac{8}{5}\sqrt{221} + \frac{1}{5}\sqrt{25t^2 - 880t + 14144}, \frac{5t-88}{\sqrt{25t^2-880t+14144}}, \frac{t}{8} - \frac{11}{10}\right), \\ x_+(t) = \left(1 - \frac{8}{5}\sqrt{181} + \frac{1}{5}\sqrt{25t^2 - 720t + 11584}, \frac{5t-72}{\sqrt{25t^2-720t+11584}}, \frac{t}{8} + \frac{9}{10}\right). \end{cases}$$

In this case the chains needed in the proof of Corollary 3.2 are reduced to ordinary iteration sequences $(G^n x_\pm)$, where $G = (G_1, G_2, G_3)$ is defined by (3.12). Calculating the iterations $G^n x_-$, it turns out that $G^9 x_- = G^{10} x_-$. Thus $u_* = G_1^9 x_-$ is a least solution of (3.16). Similarly, one can show that $u^* = G_1^{15} x_+ = G_1^{16} x_+$ is a greatest solution of (3.16). The exact expressions of these solutions are

$$\begin{cases} u_*(t) = -\frac{699}{2000} - \frac{8}{5}\sqrt{109} + \frac{1}{5}\sqrt{25t^2 - 240t + 6976}, \\ u^*(t) = \frac{177}{1000} - \frac{4}{5}\sqrt{401} + \frac{1}{5}\sqrt{25t^2 + 40t + 6416}. \end{cases}$$

4. Existence results for second-order implicit boundary value problems

This section is devoted to the study of the implicit phi-Laplacian boundary value problem (BVP),

$$\begin{cases} Lu(t) := -\frac{d}{dt}(p(t)\phi(u'(t))) = f(t, u, u', Lu) \quad \text{for a.e. } t \in J := (a, b), \\ \lim_{t\to a+} p(t)\phi(u'(t)) = c(u, u', Lu), \quad \lim_{t\to b-} u(t) = d(u, u', Lu), \end{cases} \tag{4.1}$$

where $-\infty \leqslant a < b \leqslant \infty$, $p \in L^1_{\mathrm{loc}}(J)$, $\phi : I \to \mathbb{R}$, $f : J \times L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J) \to \mathbb{R}$ and $c, d : L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J) \to \mathbb{R}$.

Denote

$$Z := \left\{ h \in X \;\middle|\; \int_r^{b-} h(t)\,dt = \lim_{s\uparrow b} \int_r^s h(t)\,dt \text{ is finite for some } r \in J \right\}, \tag{4.2}$$

where X is defined by (3.3). Assuming that $L^1_{\mathrm{loc}}(J)$ and Z are ordered a.e. pointwise, we shall show that the BVP (4.1) has extremal solutions in the pointwise ordered set Y defined in (3.2) if the functions p, ϕ, f, c and d satisfy the following hypotheses:

- (ϕ) ϕ is an increasing homeomorphism from an open interval I of $\mathbb{R}$ onto $\mathbb{R}$.
- (ϕp) p is a.e. positive-valued, and $\left|\int_t^{b-} \phi^{-1}\left(\frac{K}{p(s)}\right) ds\right| < \infty$ for all $t \in J$ and $K \in \mathbb{R}$.
- (f_0) $f(\cdot, u, v, w)$ is Lebesgue measurable and $Z \ni h_- \leqslant f(\cdot, u, v, w) \leqslant h_+ \in Z$ for all $u, v, w \in L^1_{\mathrm{loc}}(J)$.
- (f_1) There exists a $\lambda \geqslant 0$ such that $f(\cdot, u_1, v_1, w_1) + \lambda w_1 \leqslant f(\cdot, u_2, v_2, w_2) + \lambda w_2$ whenever $u_i, v_i, w_i \in L^1_{\mathrm{loc}}(J)$, $i = 1, 2$, $u_1 \leqslant u_2$, $v_1 \geqslant v_2$ and $w_1 \leqslant w_2$.
- (c_1) $c_\pm \in \mathbb{R}$, and $c_- \leqslant c(u_2, v_2, w_2) \leqslant c(u_1, v_1, w_1) \leqslant c_+$ whenever $u_i, v_i, w_i \in L^1_{\mathrm{loc}}(J)$, $i = 1, 2$, $u_1 \leqslant u_2$, $v_1 \geqslant v_2$ and $w_1 \leqslant w_2$.

(d_1) $d_\pm \in \mathbb{R}$, and $d_- \leqslant d(u_1, v_1, w_1) \leqslant d(u_2, v_2, w_2) \leqslant d_+$ whenever $u_i, v_i, w_i \in L^1_{\mathrm{loc}}(J)$, $i = 1, 2$, $u_1 \leqslant u_2$, $v_1 \geqslant v_2$ and $w_1 \leqslant w_2$.

The method is the same as in Section 3, that is, we shall first convert the BVP (4.1) to a system of three equations, and then apply the fixed point result of Lemma 2.3.

Lemma 4.1. *Assume that the hypotheses* (ϕ) *and* ($p\phi$) *hold, and that* $f(\cdot, u, v, w) \in Z$ *for all* u, v, $w \in L^1_{\mathrm{loc}}(J)$. *Then* u *is a solution of the IVP* (4.1) *in* Y, *defined by* (3.2) *if and only if* $(u, u', Lu) = (u, v, w)$, *where* (u, v, w) *is a solution of the system*

$$\begin{cases} u(t) = d(u, v, w) - \int_t^{b-} v(s)\,ds, & t \in J, \\ v(t) = \phi^{-1}\big(\frac{1}{p(t)}\big[c(u, v, w) - \int_{a+}^t w(s)\,ds\big]\big), & t \in J, \\ w(t) = f(t, u, v, w) & \text{for a.e. } t \in J, \end{cases} \tag{4.3}$$

in $L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J)$.

Proof. Assume that u is a solution of (4.1) in Y, and denote

$$w(t) = Lu(t) = -\frac{d}{dt}\big(p(t)\phi\big(u'(t)\big)\big), \qquad v(t) = u'(t), \quad t \in J. \tag{4.4}$$

The differential equation and the second initial condition of (4.1), the definition (3.2) of Y and notations (4.4) ensure that first and third equations of (4.3) hold, and that

$$\int_r^s w(t)\,dt = -\int_r^s \frac{d}{dt}\big(p(t)\phi\big(v(t)\big)\big)\,dt = p(r)\phi\big(v(r)\big) - p(s)\phi\big(v(s)\big),$$
$$a < r \leqslant s < b.$$

This result and the first initial condition of (4.1) imply that the second equation of (4.3) holds.

Conversely, let (u, v, w) be a solution of the system (4.3) in $L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J)$. The first equation of (4.3) implies that $v = u'$, that u is locally absolutely continuous, and that the second initial condition of (4.1) holds. Since $v = u'$, it follows from the second equation of (4.3) that

$$p(t)\phi\big(u'(t)\big) = c(u, u', w) - \int_{a+}^t w(s)\,ds, \quad t \in J. \tag{4.5}$$

This equation implies that $p \cdot (\phi \circ u')$ is locally absolutely continuous, and thus $u \in Y$. It follows from (4.5) by differentiation that

$$w(t) = -\frac{d}{dt} p(t)\phi\big(u'(t)\big) = Lu(t) \quad \text{for a.e. } t \in J. \tag{4.6}$$

This result and (4.5) imply that the first initial condition of (4.1) holds. The validity of the differential equation of (4.1) is a consequence of the third equation of (4.3), Eq. (4.6), and the fact that $v = u'$. $\square$

The main existence result for the BVP (4.1) reads as follows.

Theorem 4.1. *Assume that the hypotheses* (ϕ), (ϕp), (f_0), (f_1), (c_1) *and* (d_1) *hold. Then the BVP* (4.1) *has least and greatest solutions in* Y*, and they are increasing with respect to* f *and* d *and decreasing with respect to* c*.*

Proof. Assume that $P = L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J)$ is ordered by

$$(u_1, v_1, w_1) \preccurlyeq (u_2, v_2, w_2) \quad \text{if and only if } u_1 \leqslant u_2,\ v_1 \geqslant v_2 \text{ and } w_1 \leqslant w_2. \tag{4.7}$$

The relations

$$\begin{cases} x_-(t) := \big(d_- - \int_t^{b-} \phi^{-1}\big(\frac{1}{p(s)}\big[c_+ - \int_{a+}^s h_-(\tau)\,d\tau\big]\big)\,ds, \\ \qquad\qquad \phi^{-1}\big(\frac{1}{p(t)}\big[c_+ - \int_{a+}^t h_-(s)\,ds\big]\big), h_-(t)\big), \\ x_+(t) := \big(d_+ - \int_t^{b-} \phi^{-1}\big(\frac{1}{p(s)}\big[c_- - \int_{a+}^s h_+(\tau)\,d\tau\big]\big)\,ds, \\ \qquad\qquad \phi^{-1}\big(\frac{1}{p(t)}\big[c_- - \int_{a+}^t h_+(s)\,ds\big]\big), h_+(t)\big), \end{cases} \tag{4.8}$$

define functions $x_\pm \in P$, and $x_- \preccurlyeq x_+$. Moreover, it is easy to show, by applying the given hypotheses, that the relations

$$\begin{cases} G_1(u, v, w)(t) = d(u, v, w) - \int_t^{b-} v(s)\,ds, & t \in J, \\ G_2(u, v, w)(t) = \phi^{-1}\big(\frac{1}{p(t)}\big[c(u, v, w) - \int_{a+}^t w(s)\,ds\big]\big), & t \in J, \\ G_3(u, v, w)(t) = \frac{f(t,u,v,w)+\lambda w(t)}{1+\lambda}, & t \in J, \end{cases} \tag{4.9}$$

define an increasing mapping $G = (G_1, G_2, G_3) : [x_-, x_+] \to [x_-, x_+]$.

Let W be a well-ordered chain in ran G. The sets $W_1 = \{u \mid (u, v, w) \in W\}$ and $W_3 = \{w \mid (u, v, w) \in W\}$ are well-ordered, $W_2 = \{v \mid (u, v, w) \in W\}$ is inversely well-ordered, and all three are order-bounded in $L^1_{\mathrm{loc}}(J)$. It then follows from Lemma 2.3 that the supremums of W_1 and W_3 and an infimum of W_2 exist in $L^1_{\mathrm{loc}}(J)$. Obviously, $(\sup W_1, \inf W_2, \sup W_3)$ is a supremum of W in $(P, \preccurlyeq)$. Similarly one can show that each inversely well-ordered chain of ran G has an infimum in $(P, \preccurlyeq)$.

The above proof shows that the operator $G = (G_1, G_2, G_3)$ defined by (4.9) satisfies the hypotheses of Lemma 2.2, whence G has a least fixed point $x_* = (u_*, v_*, w_*)$ and a greatest fixed point $x^* = (u^*, v^*, w^*)$. It follows from (4.9) that (u_*, v_*, w_*) and (u^*, v^*, w^*) are solutions of the system (4.3). According to Lemma 4.1 u_* and u^* belong to Y and are solutions of the IVP (4.1).

To prove that u_* and u^* are least and greatest of all solutions of (4.1) in Y, let $u \in Y$ be a solution of (4.1). In view of Lemma 4.1, $(u, v, w) = (u, u', Lu)$ is a solution of the system (4.3). Applying the hypotheses (f_1), (c_1) and (d_1) it is easy to show that $x = (u, v, w) \in [x_-, x_+]$, where $x_\pm$ are defined by (4.8). Thus $x = (u, v, w)$ is a fixed point of $G = (G_1, G_2, G_2) : [x_-, x_+] \to [x_-, x_+]$, defined by (4.9). Because $x_* = (u_*, v_*, w_*)$ and $x^* = (u^*, v^*, w^*)$ are least and greatest fixed points of G, then $(u_*, v_*, w_*) \preccurlyeq (u, v, w) \preccurlyeq (u^*, v^*, w^*)$. In particular, $u_* \leqslant u \leqslant u^*$, whence u_* and u^* are least and greatest of all solutions of the IVP (4.1).

The last assertion is an easy consequence of the last conclusion of Lemma 2.2 and the definition (4.9) of $G = (G_1, G_2, G_3)$. □

If we replace the hypothesis (f_0) by the following hypothesis:

(f_2) $f(\cdot, u, v, w) \in Z$ for all $u, v, w \in P = L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J) \times L^1_{\mathrm{loc}}(J)$, and there exist $x_\pm = (u_\pm, v_\pm, w_\pm) \in P$ such that $x_- \preccurlyeq x_+$, $x_- \preccurlyeq Gx_-$ and $Gx_+ \preccurlyeq x_+$, where $G = (G_1, G_2, G_3)$ is defined by (4.9),

we get the following result.

Proposition 4.1. *Assume that the hypotheses* (ϕ), (ϕp), (f_1), (f_2), (c_1) *and* (d_1) *hold. Then the BVP* (4.1) *has a least and a greatest solution in* $\{u \in Y \mid u_- \leqslant u \leqslant u_+\}$.

As a special case, we obtain an existence result for the BVP,

$$\begin{cases} -\frac{d}{dt}(p(t)\phi(u'(t))) = g(t, u(t), u'(t), -\frac{d}{dt}(p(t)\phi(u'(t)))) & \text{for a.e. } t \in J, \\ \lim_{t\to a+} p(t)\phi(u'(t)) = c, \quad \lim_{t\to b-} u(t) = d. \end{cases} \tag{4.10}$$

Corollary 4.1. *Let the hypotheses* (ϕ) *and* (ϕp) *hold, and let* $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfy the following hypotheses*:

(g_0) $g(\cdot, u(\cdot), v(\cdot), w(\cdot))$ *is Lebesgue measurable and* $h_- \leqslant g(\cdot, u(\cdot), v(\cdot), w(\cdot)) \leqslant h_+$ *for all* $u,\ v,\ w \in L^1_{\mathrm{loc}}(J)$ *and for some* $h_\pm \in Z$.

(g_1) *There exists a* $\lambda \geqslant 0$ *such that* $g(t, x_1, y_1, z_1) + \lambda z_1 \leqslant g(t, x_2, y_2, x_2) + \lambda z_2$ *for a.e.* $t \in J$ *and whenever* $x_1 \leqslant x_2$, $y_1 \geqslant y_2$ *and* $z_1 \leqslant z_2$ *in* $\mathbb{R}$.

Then the BVP (4.10) *has for each choice of* $c,\ d \in \mathbb{R}$ *least and greatest solutions in* Y. *Moreover, these solutions are increasing with* g *and* d *and decreasing with respect to* c.

Proof. If $c,\ d \in \mathbb{R}$, the BVP (4.10) is reduced to (4.1) when we define

$$\begin{cases} f(t, u, v, w) = g(t, u(t), v(t), w(t)), \quad t \in J,\ u, v, w \in L^1_{\mathrm{loc}}(J), \\ c(u, v, w) \equiv c, \quad d(u, v, w) \equiv d,\ u, v, w \in L^1_{\mathrm{loc}}(J). \end{cases} \tag{4.11}$$

The hypotheses (g_0) and (g_1) imply that f satisfies the hypotheses (f_0) and (f_1). The hypotheses (c_1) and (d_1) is also valid, whence (4.1), with f, c and d defined by (4.11), and hence also (4.10), has by Theorem 4.1 least and greatest solutions. The last assertion follows from the last assertion of Theorem 4.1. □

In the case when ϕ is defined by

$$\phi(x) = \frac{x}{\sqrt{1 - x^2}}, \quad x \in (-1, 1),$$

the operator G given by (4.9) can be rewritten as

$$\begin{cases} G_1(u,v,w)(t) = d(u,v,w) - \int_t^{b-} v(s)\,ds, & t \in J, \\ G_2(u,v,w)(t) = \frac{c(u,v,w) - \int_{a+}^t w(s)\,ds}{\sqrt{p^2(t) + (c(u,v,w) - \int_{a+}^t w(s)\,ds)^2}}, & t \in J, \\ G_3(u,v,w)(t) = \frac{f(t,u,v,w) + \lambda w(t)}{1+\lambda}, & t \in J. \end{cases} \tag{4.12}$$

This formula shows that $-1 \leqslant G_2(u,v,w)(t) \leqslant 1$ whenever $G_2(u,v,w)$ is defined. Thus we have the following result.

Corollary 4.2. *If $-\infty \leqslant a < b < \infty$, the IVP*

$$\begin{cases} Lu(t) := -\frac{d}{dt}\Big(\frac{p(t)u'(t)}{\sqrt{1-u'(t)^2}}\Big) = f(t,u,u',Lu) \quad \text{a.e. in } J = (a,b), \\ \lim_{t\to a+} \frac{p(t)u'(t)}{\sqrt{1-u'(t)^2}} = c(u,u',Lu), \quad u(b) = d(u,u',Lu), \end{cases} \tag{4.13}$$

has least and greatest solutions if $p \in L^1_{\mathrm{loc}}(J)$ is positive-valued, if the hypotheses (f_0), (f_1) and (d_1) hold, and if $c(u_2,v_2,w_2) \leqslant c(u_1,v_1,w_w)$ in $\mathbb{R}$ whenever $u_1 \leqslant u_2$ and $v_1 \geqslant v_2$ and $w_1 \leqslant w_2$ in $L^1_{\mathrm{loc}}(J)$.

Remark 4.1. If $h(t) = \frac{1}{t}\sin\frac{1}{t}$, $t \in J$, then h and the function $f(\cdot,u,u')$ defined in (4.13) belong to Z, but not in $L^1(J)$.

Example 4.1. Determine least and greatest solutions of the BVP,

$$\begin{cases} -\frac{d}{dt}\Big(\frac{tu'(t)}{\sqrt{1-u'(t)^2}}\Big) = \frac{2t-1}{8} + \frac{[100\tan^{-1}(\int_1^2 (u(s)-u'(s)+Lu(s))\,ds)]}{200} \quad \text{a.e. in } (0,3), \\ \lim_{t\to 0+} \frac{tu'(t)}{\sqrt{1-u'(t)^2}} = 0, \quad u(3) = \frac{[100\tanh((u(1)-u'(1)+Lu(1)))4]}{1000}. \end{cases} \tag{4.14}$$

Solution. (4.14) is a special case of (4.13) when $p(t) = t$. It is also easy to see that the hypotheses of Corollary 4.2 are satisfied. Thus (4.14) has extremal solutions. The functions $x_\pm$ defined by (4.8) can be calculated, and one obtains

$$\begin{cases} x_-(t) = \Big(9 - \sqrt{t^2 - 18t + 145}, \frac{t-9}{\sqrt{t^2-18t+145}}, \frac{2t-9}{8}\Big), \\ x_+(t) = \Big(1 + 2\sqrt{41} - \sqrt{t^2 + 14t + 113}, \frac{t+7}{\sqrt{t^2+14t+113}}, \frac{2t+7}{8}\Big). \end{cases}$$

In this case the chains needed in the proof of Corollary 4.2 are reduced to ordinary iteration sequences $(G^n x_\pm)$, where G is defined by (4.9). Calculating the iterations $G^n x_-$, it turns out that $G^8 x_- = G^9 x_-$. Thus $u_* = G_1^8 x_-$ is a least solution of (4.14). Similarly, one can show that $u^* = G_1^4 x_+ = G_1^5 x_+$ is a greatest solution of (4.14). The exact expressions of these solutions are

$$\begin{cases} u_*(t) = -\frac{57}{250} + \frac{1}{25}\sqrt{41369} - \frac{1}{25}\sqrt{625t^2 - 5600t + 52544}, \\ u^*(t) = \frac{247}{1000} + \frac{1}{25}\sqrt{68561} - \frac{1}{25}\sqrt{625t^2 + 4700t + 48836}. \end{cases}$$

Remarks 4.3. Problems of the form (2.1), (3.1) and (4.1) include many kinds of special types. For instance, they can be

- implicit because both differential equations and initial/boundary conditions depend on the differential operator Lu;
- singular, because a case $\lim_{t\to a+} p(t) = 0$ is allowed;
- functional, because the functions c, d and f may depend functionally on u, Lu and/or u';
- discontinuous, since the dependencies of c, d and f on u, u' and Lu can be discontinuous;
- problems on unbounded intervals, because cases $a = -\infty$ and/or $b = \infty$ are included;
- p-Laplacian when ϕ is defined by (3.11).

Explicit problems, i.e., cases where neither differential equations nor initial/boundary conditions depend on the differential operator Lu, are considered in [8]. As for uniqueness results for phi-Laplacian initial and boundary value problems see, e.g., [4,7]. In [1–3,9,10] existence results are introduced for these problems.

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